Natural Resource Economics

Notes and Problems

JON M. CONRAD

Cornell University

and

COLIN W. CLARK

University of British Columbia

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Chapter 1

Resource allocation and optimization

Economics has been defined as the science of allocating scarce resources among competing ends. Much of the microeconomic theory encountered in a first semester graduate course is concerned with the *static* allocation problem faced by firms and consumers. Techniques of constrained optimization, in particular the method of Lagrange multipliers, are employed in developing the theory of the firm and the consumer.

The optimal harvest of renewable resources or extraction of exhaustible resources is inherently a *dynamic* allocation problem; that is, the firm or resource manager is concerned with the best harvest or extraction rate through time. It turns out that the method of Lagrange multipliers can be extended to intertemporal or dynamic allocation problems in a relatively straightforward fashion. This "discrete-time" extension of the method of Lagrange multipliers serves as a useful springboard to the "continuous-time" solution of dynamic allocation problems via the maximum principle. The method of Lagrange multipliers and its various extensions reduce the original optimization problem to a system of equations to be solved. Solving this system of equations, unfortunately, can often be exceedingly difficult, especially for dynamic problems. There are also technical problems concerning sufficiency conditions for the solutions so obtained (and also pertaining to the existence of a solution to the given problem). In these notes we will normally consider problems which are simple enough that these difficulties are minimized.

Before presenting the dynamic techniques we will briefly review the method of Lagrange multipliers within the context of allocating scarce resources among competing ends at a single point in time.

1.1 Constrained optimization and the method of Lagrange multipliers

In resource economics as in other fields of economics, we often encounter constrained optimization problems. The general form of such problems is

$$\begin{aligned} &\text{maximize} \quad V(x_1, \ldots, x_n) \\ &\text{subject to} \quad (x_1, \ldots, x_n) \in A \end{aligned} \tag{1.1}$$

where $V(\cdot)$ is a given *objective* (or *value*) *function* of n decision variables $x_1, \ldots, x_n$ which are required to be in some given *constraint set* A.

In the case of a *static* optimization problem, the decision variables x_i are real numbers and the constraint set A is a subset of $\mathbb{R}^n$–Euclidean n-space. For *dynamic* optimization problems, on the other hand, some (or all) of the decision variables are functions of time t (usually separated into so-called *state* variables and *control* variables). The constraint condition then typically involves the system's *dynamics*, expressed as a system of differential or difference equations. Other constraints may also be present. As before, $V(\cdot)$ is real-valued, frequently involving integration (or summation) over time. Dynamic optimization problems will be considered in Section 1.2.

1.1.1 Static optimization: no constraints

The simplest optimization problem is

$$\text{maximize} \quad V(x_1, \ldots, x_n) \tag{1.2}$$

where the decision variables are unconstrained.[1] The reader is assumed to be familiar with the first order necessary conditions

$$\frac{\partial V(\cdot)}{\partial x_i} = 0 \qquad\qquad i = 1, \ldots, n \tag{1.3}$$

By "necessary" conditions we mean that equations (1.3) must be satisfied by the maximizing values of $x_1, \ldots, x_n$. The conditions (1.3) are not sufficient conditions for a maximum, however (they also pertain to minimal solutions and to values x_i which are neither maxima nor minima). We will not attempt to delineate sufficient conditions in these notes [see Intriligator (1971, p 26)], since such conditions often are complicated and of very limited practical use (but popular with economics professors). If the objective function $V(\cdot)$ is known to be concave, the necessary conditions (1.3) are also sufficient.[2] Note that (1.3) constitutes a system of n possibly nonlin-

---

[1] We shall assume throughout that $V(\cdot)$ is a smooth function; that is, all required partial derivatives exist.

[2] The function $V(X)$ is said to be concave if

$$V(\alpha \bar{X} + (1-\alpha)\tilde{X}) \geq \alpha V(\bar{X}) + (1-\alpha)V(\tilde{X})$$

for all $\bar{X} = (\bar{x}_1, \ldots, \bar{x}_n)$, $\tilde{X} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ and $0 \leq \alpha \leq 1$.

ear equations in n unknowns $x_1, \ldots, x_n$. Thus the optimization problem has been "reduced" to the solution of n equations. Unfortunately, solving the system in practice may be almost as difficult as the original optimization problem. Numerical algorithms for the solution of such systems are available in computing centers, but do not always work (see Section 1.6.2 for an example). Frequently a direct optimization algorithm (based on a direct "search" of the feasible set) will outperform any method based on the first order necessary conditions.

Nevertheless, insight into economics is often obtained from the necessary conditions without actually solving them explicitly. For example, if $V(\cdot)$ is a net benefit function, the statement "marginal net benefit of each input x_i must equal zero" is equivalent to (1.3) and carries economic significance.

1.1.2 Static optimization: equality constraints

Consider next the constrained problem

$$\begin{aligned} &\text{maximize} \quad V(x,y,z) \\ &\text{subject to} \quad G(x,y,z) = c \end{aligned} \tag{1.4}$$

where for simplicity we have only three decision variables x, y, and z. The equation $G(\cdot) = c$, where c is a known constant, determines a constraint set in x, y, z space, which is in fact a surface, which we will denote by S_G. The problem, then, is to determine the largest value of the function $V(x,y,z)$ for points (x,y,z) on the surface S_G.

One approach to this problem is first to solve the constraint equation $G(\cdot) = c$ for one of the variables, say $z = h(x,y)$. The constrained problem in three variables may be replaced by the unconstrained two variable problem

$$\text{maximize} \quad V(x,y,h(x,y)) \tag{1.5}$$

with first order necessary conditions

$$\left.\begin{aligned} \frac{\partial V}{\partial x} + \frac{\partial V}{\partial z}\frac{\partial h}{\partial x} = 0 \\ \frac{\partial V}{\partial y} + \frac{\partial V}{\partial z}\frac{\partial h}{\partial y} = 0 \end{aligned}\right\} \tag{1.6}$$

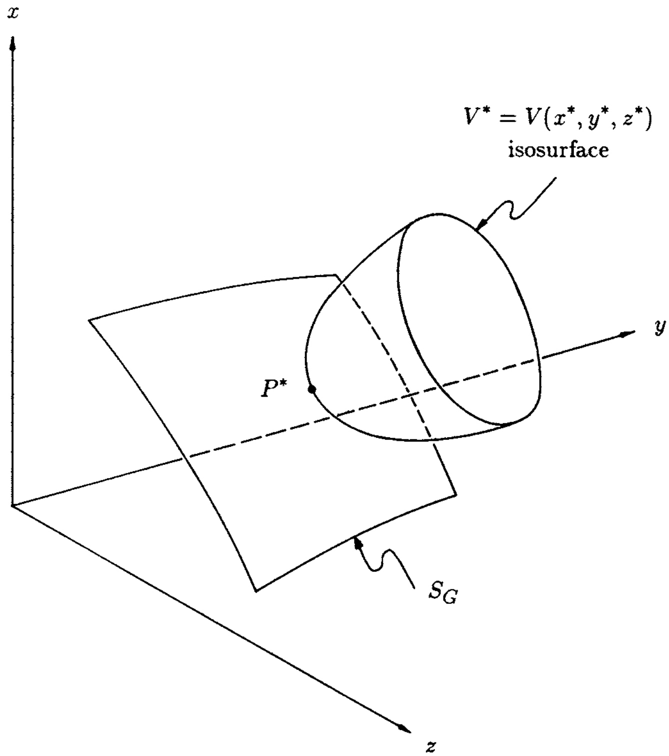


Figure 1.1 The tangency criterion: if $V(\cdot)$ is maximized on S_G at P^* then the V^* isosurface through P^* is tangent to S_G.

Differentiating the constraint equation implicitly implies

$$\left.\begin{aligned} \frac{\partial G}{\partial x} + \frac{\partial G}{\partial z}\frac{\partial h}{\partial x} = 0 \\ \frac{\partial G}{\partial y} + \frac{\partial G}{\partial z}\frac{\partial h}{\partial y} = 0 \end{aligned}\right\} \tag{1.7}$$

and Equations (1.6) can therefore be written in the form

$$\left.\begin{aligned} V_x G_z - V_z G_x = 0 \\ V_y G_z - V_z G_y = 0 \end{aligned}\right\} \tag{1.8}$$

where V_x is shorthand for $\partial V/\partial x$, etc.

Equations (1.8) can be obtained from an alternative geometrical derivation which gives important mathematical and economic insight. Figure 1.1 shows the solution to (1.4) as it would appear in x, y, z space. The constraint surface is shown as S_G. Suppose $P^* = (x^*, y^*, z^*)$ is the point on S_G for which $V(x, y, z)$ attains its maximum. Consider also the isosurface of $V(\cdot)$ passing through P^*; this is the surface

$$V(x, y, z) = V(x^*, y^*, z^*) \tag{1.9}$$

After a moment's reflection, one would conclude that $V(x^*, y^*, z^*)$ must be tangent to the constraint surface S_G. If it were not, it would either cut through S_G or not touch it at all. In the latter case the constraint is not satisfied. In the former case there would be a projection of the isosurface on S_G and there would be points on S_G lying inside and outside the projection.[3] Since $V = V^*$ on the projection we must have $V > V^*$ on one side (say inside) of the projection and $V < V^*$ on the other side (outside) of the projection. But V^* was by assumption the maximum of $V(\cdot)$ on S_G. Therefore no points of S_G can have a value $V > V^*$, which would be a contradiction. The conclusion: The maximizing isosurface must be tangent to S_G at P^*.

Recall now from calculus that the *gradient vector*

$$\overrightarrow{\nabla} V = (V_x, V_y, V_z) \tag{1.10}$$

is always perpendicular (normal) to the isosurface $V(\cdot) =$ constant at any given point. Two isosurfaces passing through a point P^* are therefore tangent at P^* if and only if their gradient vectors have the same direction. This means that $\overrightarrow{\nabla} V = \lambda \overrightarrow{\nabla} G$ for some $\lambda \neq 0$. This vector equation means, in turn, that

$$\left.\begin{aligned} V_x &= \lambda G_x \\ V_y &= \lambda G_y \\ V_z &= \lambda G_z \end{aligned}\right\} \tag{1.11}$$

at point P^*.

Note that, by dividing pairs of equations, (1.11) reduces to the necessary conditions (1.8). Conversely (1.8) implies (1.11).[4] Equations (1.11), however, have an appealing symmetry lacking in (1.8). They also generalize in a nice way, as we shall see.

---

[3] The projection on an isosurface which cuts through S_G might appear as a bent circle or ellipse in Figure 1.1. Think of the "nose" of some other isosurface lying on the "other side" of S_G and the projection as a closed contour on S_G.

[4] From (1.8) we have $V_x/G_x = V_y/G_y = V_z/G_z$. Call the common value $\lambda = V_z/G_z$.

1.1.3 Lagrange multipliers

It is customary to rewrite Equations (1.11) in the form

$$\left.\begin{aligned} V_x - \lambda G_x = 0 \\ V_y - \lambda G_y = 0 \\ V_z - \lambda G_z = 0 \end{aligned}\right\} \tag{1.12}$$

If we define the expression

$$L = V(x, y, z) - \lambda[G(x, y, z) - c] \tag{1.13}$$

then Equations (1.12) are also obtained as

$$L_x = L_y = L_z = 0 \tag{1.14}$$

The expression L is called the *Lagrangian* associated with the original constrained optimization problem (1.4). The number λ is referred to as a *Lagrange multiplier.* Thus, the critical observation in the development of the *method of Lagrange multipliers* was that differentiation of the Lagrangian expression would lead to the same first order necessary conditions as obtained in the simple "constraint substitution" technique used to transform an equality constrained problem into an unconstrained problem [i.e., going from problem (1.4) to problem (1.5)].

Consider now the general optimization problem with multiple equality constraints

$$\begin{aligned} \text{maximize} \quad & V(x_1 \ldots, x_n) \\ \text{subject to} \quad & G_j(x_1, \ldots, x_n) = c_j, \quad j = 1, \ldots, m \end{aligned} \tag{1.15}$$

The associated Lagrangian is

$$L = V(\cdot) - \sum_{j=1}^{m} \lambda_j [G_j(\cdot) - c_j] \tag{1.16}$$

Note that each of the m constraints gives rise to a Lagrange multiplier, λ_j. By the same sort of tangency argument as before it can be shown that the following equations are necessary conditions for $x_1, \ldots, x_n$ to be a solution to the above optimization problem

$$\frac{\partial L}{\partial x_i} = 0 \qquad i = 1, \ldots, n \tag{1.17}$$

Explicitly, these equations are

$$\frac{\partial V}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial G_j}{\partial x_i} = 0 \qquad i = 1, \ldots, n \tag{1.18}$$

We might also note that

$$\frac{\partial L}{\partial \lambda_j} = -G_j(\cdot) + c_j = 0 \qquad j = 1, \ldots, m \tag{1.19}$$

and that when taken together, Equations (1.18) and (1.19) constitute a system of $n+m$ equations in $n+m$ unknowns: $x_1, \ldots, x_n$; $\lambda_1, \ldots, \lambda_m$. In principle this system should have at most a finite number of solutions, one of which will be the solution to our original optimization problem. In practice, as we noted earlier, solving this system of equations may be difficult indeed.

Consider the following example

$$\begin{aligned} \text{maximize} \quad & 2x - 3y + z \\ \text{subject to} \quad & x^2 + y^2 + z^2 = 9 \end{aligned}$$

The Lagrangian for this problem is

$$L = 2x - 3y + z - \lambda(x^2 + y^2 + z^2 - 9)$$

with the first order necessary conditions

$$\begin{aligned} \frac{\partial L}{\partial x} &= 2 - 2\lambda x = 0 \\ \frac{\partial L}{\partial y} &= -3 - 2\lambda y = 0 \\ \frac{\partial L}{\partial z} &= 1 - 2\lambda z = 0 \\ \frac{\partial L}{\partial \lambda} &= -x^2 - y^2 - z^2 + 9 = 0 \end{aligned}$$

The first and second and first and third equations imply $y = -3x/2$ and $z = x/2$; which upon substitution into the constraint equation yields

$$x^2 + \left(\frac{-3x}{2}\right)^2 + \left(\frac{x}{2}\right)^2 = 0$$

which may be solved for $x = \pm 3\sqrt{2/7}$ leading to two solutions

$$x_1 = 3\sqrt{2/7} = 1.60 \qquad y_1 = -9/\sqrt{14} = -2.41 \qquad z_1 = 3/\sqrt{14} = 0.80$$

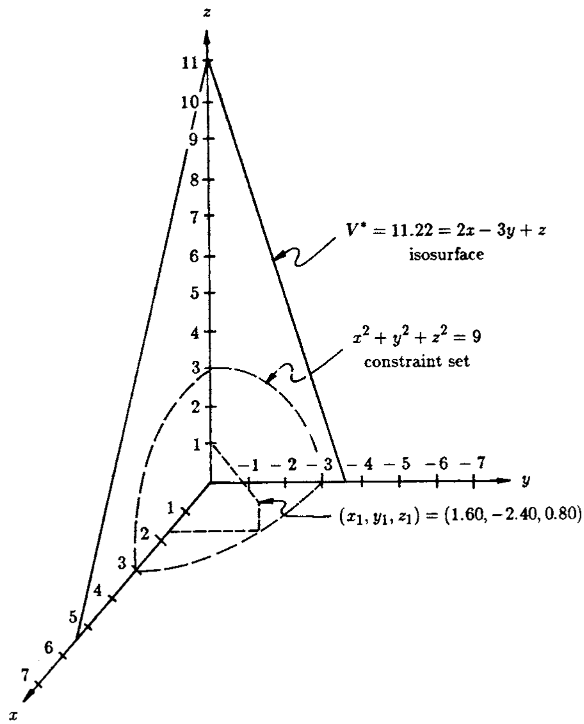


Figure 1.2 Depiction of the problem: maximize $2x - 3y + z$, subject to $x^2 + y^2 + z^2 = 9$, which has a solution at $(x_1, y_1, z_1) = (1.60, -2.41, 0.80)$.

and

$$x_2 = -3/\sqrt{2/7} = -1.60 \quad y_2 = 9/\sqrt{14} = 2.41 \quad z_2 = -3/\sqrt{14} = -0.80$$

We note, however, that

$$V(x_1, y_1, z_1) = 42/\sqrt{14} = 11.22$$
$$V(x_2, y_2, z_2) = -42/\sqrt{14} = -11.22$$

Thus, the maximizing point is (x_1, y_1, z_1), while (x_2, y_2, z_2) is a minimizing point. The problem and solution are depicted in Figure 1.2.

1.1.4 Economic interpretation

The Lagrange multipliers λ_j were not part of the original optimization problem. In the above example, for instance, we eliminated λ and then forgot about it. But Lagrange multipliers do have an important economic interpretation.

Clearly the solution to the general problem (1.15) will depend upon the values of the parameters $c_1, \ldots, c_m$ in the constraint equations $G_j(\cdot) = c_j$, $j = 1, \ldots, m$. In fact we may explicitly state this dependence as

$$x_i^* = x_i(c_1, \ldots, c_m) \tag{1.20}$$

If the optimal values for the decision variables depend on the parameters then so does the value of the objective function. Consider

$$\frac{\partial V}{\partial c_k} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \frac{\partial x_i}{\partial c_k} \qquad k = 1, \ldots, m \tag{1.21}$$

From Equations (1.18) we know

$$\frac{\partial V}{\partial x_i} = \sum_{j=1}^{m} \lambda_j \frac{\partial G_j}{\partial x_i} \tag{1.22}$$

and thus

$$\frac{\partial V}{\partial c_k} = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \lambda_j \frac{\partial G_j}{\partial x_i} \right) \frac{\partial x_i}{\partial c_k} \qquad k = 1, \ldots, m \tag{1.23}$$

Finally, differentiating the constraint equation $G_j(\cdot) = c_j$ with respect to c_k, we have

$$\sum_{i=1}^{n} \frac{\partial G_j}{\partial x_i} \frac{\partial x_i}{\partial c_k} = \delta_{j,k} = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases} \tag{1.24}$$

Hence we find that

$$\frac{\partial V}{\partial c_k} = \sum_{j=1}^{m} \lambda_j \delta_{j,k} = \lambda_k \tag{1.25}$$

The Lagrange multiplier λ_k thus equals the incremental change in value from an incremental change in the constraint parameter c_k. In other words, λ_k represents the marginal value of relaxing the kth constraint. If c_k represents the available supply of some input or resource, then λ_k represents the "price" (or value) of the input in terms of V; hence λ_k is often called the *shadow price* of the input c_k.

Consider a resource-based economy which can allocate labor (L) to harvest timber (T) or fish (F). Assume that the economy exports both timber and fish, facing constant world prices denoted P_T and P_F, respectively. The transformation curve, relating technically efficient combinations of timber, fish, and labor, is given by

$$G(T, F; L) = T^2 + F^2/4 - L = 0$$

Suppose $P_T =$ \$500/metric ton, $P_F =$ \$1,000/metric ton and $L = 1700$ is the number of available hours of labor to be allocated between harvesting timber or fish. The static optimization problem seeks to maximize the value of harvest subject to the transformation function; that is

$$\begin{aligned} \text{maximize} \quad & V = 500T + 1,000F \\ \text{subject to} \quad & T^2 + F^2/4 - 1700 = 0 \end{aligned}$$

The Lagrangian expression may be written as

$$L = 500T + 1,000F - \lambda(T^2 + F^2/4 - 1700)$$

and has first order necessary conditions which require

$$\frac{\partial L}{\partial T} = 500 - 2\lambda T = 0$$

$$\frac{\partial L}{\partial F} = 1,000 - 0.5\lambda F = 0$$

and

$$\frac{\partial L}{\partial \lambda} = -T^2 - F^2/4 + 1700 = 0$$

Taking the ratio of the first two equations to eliminate λ implies $F = 8T$. Substituting this expression for F into the transformation function yields

$$\begin{aligned} T^2 + 64T^2/4 &= 1700 \\ T^2 &= 100 \end{aligned}$$

and

$$T = 10 \qquad F = 80 \qquad \lambda = 25$$

Thus, the economy should allocate the available labor so as to produce 10 metric tons of timber and 80 metric tons of fish. The marginal value (shadow price) of an additional unit of labor is \$25/hour.[5]

[5] A check of the appropriate second order conditions would reveal $T = 10$, $F = 80$, $\lambda = 25$ to be a maximum. Note: L is concave in T and F.

1.1.5 Static optimization with inequality constraints

Next let us consider the problem

$$\begin{aligned} \text{maximize} \quad & V(x,y,z) \\ \text{subject to} \quad & G(x,y,z) \le c \end{aligned} \tag{1.26}$$

The constraint set A now consists of all points lying either on the surface S_G or one particular side of S_G. There are just two possibilities: (a) the optimizing point (x,y,z) lies on one side of S_G satisfying the strict inequality $G(x,y,z) < c$ and $\partial V/\partial x = \partial V/\partial y = \partial V/\partial z = 0$, or (b) the optimizing point (x,y,z) lies on S_G, satisfying the equality $G(x,y,z) = c$ in which case the Lagrangian conditions apply and $\partial L/\partial x = \partial L/\partial y = \partial L/\partial z = 0$, where $L = V(\cdot) - \lambda[G(\cdot) - c]$.

The two cases can be combined into a single condition called the *Kuhn–Tucker condition*, which is a necessary condition, and may be written as

$$\begin{gathered} \frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial z} = 0 \\ \lambda \begin{cases} = 0 & \text{if } G(\cdot) < c \\ \ge 0 & \text{if } G(\cdot) = c \end{cases} \end{gathered} \tag{1.27}$$

The student should check that this indeed covers cases (a) and (b) above. A frequently encountered form, equivalent to (1.27) is

$$\begin{gathered} \frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial z} = 0 \\ \lambda[G(\cdot) - c] = 0 \\ \lambda \ge 0 \end{gathered} \tag{1.28}$$

In many applications of constrained optimization in economics the decision variables are required to be *nonnegative*; i.e., $x \ge 0$, $y \ge 0$, $z \ge 0$. If problem (1.26) is amended to include nonnegativity constraints then the Kuhn–Tucker conditions become

$$\begin{gathered} x\left(\frac{\partial L}{\partial x}\right) = y\left(\frac{\partial L}{\partial y}\right) = z\left(\frac{\partial L}{\partial z}\right) = 0 \\ x \ge 0 \qquad y \ge 0 \qquad z \ge 0 \\ \lambda[G(\cdot) - c] = 0 \qquad \lambda \ge 0 \end{gathered} \tag{1.29}$$

The Kuhn–Tucker conditions are readily generalized to the case of x_1, $\ldots, x_n$ decision variables (which may be unrestricted or nonnegative in value) plus inequality constraints $G_j(x_1, \ldots, x_n) \le c_j, j = 1, \ldots, m$. The

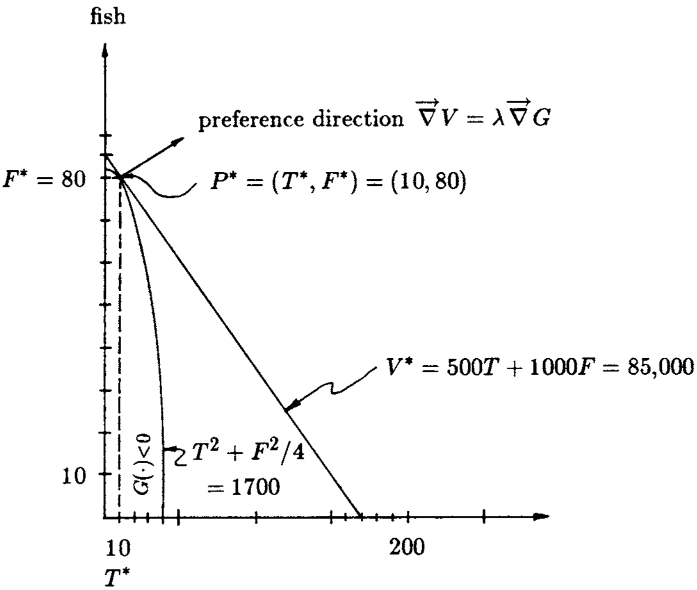


Figure 1.3 Optimal production in the timber/fish economy.

student could write out the general Kuhn–Tucker conditions as an exercise [and compare his or her version with that found in Intriligator (1971), p. 52)].

To see why $\lambda \geq 0$ let us reconsider our timber-fish economy. Suppose the transformation function were expressed as an inequality constraint; $G(T, F; L) = T^2 + F^2/4 - L \leq 0$. This constraint and the isorevenue line $V^* = 500T + 1{,}000F = 85{,}000$ are drawn in Figure 1.3. The point $P^* = (T^*, F^*) = (10, 80)$ gives the maximum revenue for $P_T = 500$, $P_F = 1{,}000$ on the constraint surface. Higher isorevenue lines would be parallel but lie to the right of V^* and thus unattainable given harvest technology and available labor.

Recall from calculus that the gradient vector, here $\vec{\nabla} V = (V_T, V_F)$, which is perpendicular to the contour $V = V^*$ also points in the direction of increasing values of V. Hence both $\vec{\nabla} V$ and $\vec{\nabla} G$ point in the same direction from P^* (namely outwards from $G(\cdot) < 0$ or inwards to $V > V^*$). Thus

$$\vec{\nabla} V = \lambda \vec{\nabla} G \tag{1.30}$$

with $\lambda > 0$.

With reference to problem (1.26) and the Kuhn–Tucker conditions as expressed in (1.27), it is possible for $\lambda = 0$ if the maximum of $V(\cdot)$ on S_G is also a *local* or *global maximum* of $V(\cdot)$ in this case $\vec{\nabla} V = 0$ so that $\lambda = 0$ in (1.30). This explains why $\lambda \geq 0$ in (1.27)–(1.29).

1.2 An extension of the method of Lagrange multipliers to dynamic allocation problems

The method of Lagrange multipliers can be employed in solving dynamic or intertemporal allocation problems and the discrete-time formulation provides a convenient introduction to control theory and the maximum principle, often presented in a continuous-time context. Let

$t = 0, 1, \ldots, T$ be the set of time periods of relevance for the dynamic allocation problem, where $t = 0$ is the present and $t = T$ is the terminal (last) period,

x_t represent a state variable, describing the system in period t,

y_t represent a control or instrument variable in period t,

$V = V(x_t, y_t, t)$ represent net economic return in period t,

$F(x_T)$ represent a final function indicating the value of alternative levels of the state variable at terminal time T, and

$x_{t+1} - x_t = f(x_t, y_t)$ be a difference equation defining the change in the state variable from period t to $(t+1)$, $t = 0, \ldots, T-1$.

The reader should note that time has been partitioned into a finite number of discrete periods, $(T+1)$ to be exact, although we can allow for an infinite horizon by letting $T \to \infty$. We will restrict ourselves to the single state, single control variable case for simplicity. The problem may be readily generalized to I state variables and J control variables. The objective function $V(\cdot)$ may have the period index t as a variable while the difference equation does not, thus $f(\cdot)$ is said to be *autonomous.*

An example of a dynamic allocation problem would be one which seeks to

$$\begin{aligned} \underset{\{y_t\}}{\text{maximize}} \quad & \sum_{t=0}^{T-1} V(x_t, y_t, t) + F(x_T) \\ \text{subject to} \quad & x_{t+1} - x_t = f(x_t, y_t) \\ & x_0 = a \quad \text{given} \end{aligned} \tag{1.31}$$

The objective in (1.31) is to maximize the sum of intermediate values plus the net value associated with the terminal state x_T. This must be done subject to the difference equation describing the change in the state variable over the horizon, assuming $x_0 = a$; that is, the initial condition is given. The problem becomes one of determining the optimal values for y_t, $t = 0, 1, \ldots, T-1$ which will, via the difference equation, imply values for x_t, $t = 1, \ldots, T$.

We can use the method of Lagrange multipliers by noting that the difference equation is a constraint equation which serves to define x_{t+1}. The Lagrangian expression may be written

$$L = \sum_{t=0}^{T-1} \{V(\cdot) + \lambda_{t+1}(x_t + f(\cdot) - x_{t+1})\} + F(\cdot) \tag{1.32}$$

where λ_{t+1} is a multiplier associated with x_{t+1}. Because there are T such constraint equations $(t = 0, \ldots, T-1)$ it is appropriate to include them within the summation operation.

With no nonnegativity constraints the first order necessary conditions require:

$$\frac{\partial L}{\partial y_t} = \frac{\partial V(\cdot)}{\partial y_t} + \lambda_{t+1}\frac{\partial f(\cdot)}{\partial y_t} = 0 \qquad t = 0, \ldots, T-1 \tag{1.33}$$

$$\frac{\partial L}{\partial x_t} = \frac{\partial V(\cdot)}{\partial x_t} + \lambda_{t+1}\left(1 + \frac{\partial f(\cdot)}{\partial x_t}\right) - \lambda_t = 0 \qquad t = 1, \ldots, T-1 \tag{1.34}$$

$$\frac{\partial L}{\partial x_T} = -\lambda_T + F'(\cdot) = 0 \tag{1.35}$$

$$\frac{\partial L}{\partial \lambda_{t+1}} = x_t + f(\cdot) - x_{t+1} = 0 \qquad t = 0, \ldots, T-1 \tag{1.36}$$

Most of the partials are straightforward with the exception of (1.34) which warrants some discussion. In taking the partial of L with respect to x_t one looks at where x_t appears in the tth term of the summation. This accounts for the first two expressions on the RHS of (1.34). If, however, one were to back up to the $(t-1)$ term one would also find a $-x_t$ premultiplied by λ_t, hence the third expression $-\lambda_t$ in (1.34).

Rewriting the first order conditions will facilitate their interpretation and put them in a form more useful when making comparisons to their continuous time counterparts. They are rewritten as

$$\frac{\partial V(\cdot)}{\partial y_t} + \lambda_{t+1}\frac{\partial f(\cdot)}{\partial y_t} = 0 \qquad t = 0, 1, \ldots, T-1 \tag{1.37}$$

$$\lambda_{t+1} - \lambda_t = -\left(\frac{\partial V(\cdot)}{\partial x_t} + \lambda_{t+1}\frac{\partial f(\cdot)}{\partial x_t}\right) \qquad t = 1, \ldots, T-1 \tag{1.38}$$

$$x_{t+1} - x_t = f(\cdot) \tag{1.39}$$

$$\lambda_T = F'(\cdot) \tag{1.40}$$

$$x_0 = a \tag{1.41}$$

Equations (1.37) will typically define a marginal condition that y_t must satisfy. However, in the dynamic allocation problem there is a term not found in static problems. In many problems in resource economics $\partial V(\cdot)/\partial y_t$ will have the interpretation of being a net marginal benefit *in period* t. This is consistent with our earlier interpretation. In the dynamic context there is a second term to be accounted for in determining the optimal y_t. This term, $\lambda_{t+1}(\partial f(\cdot)/\partial y_t)$ explicitly reflects the influence of y_t on the change in the state variable. If an increase in y_t reduces the amount of *variable* x_{t+1} then this second term reflects an inter-temporal cost, often referred to as *user cost.* Less obvious, but perhaps more important, is that in the optimal solution of the problem λ^*_{t+1} can be shown to reflect the effect that an increment in x_{t+1} would have over the *remainder* of the horizon $(t+1, \ldots, T)$. Thus there is a second cost which must be considered when undertaking an incremental action today; that is, the marginal losses that might be incurred over the remaining future.

Equation (1.38) is a difference equation which must hold through time and relates the change in the Lagrange multiplier to terms involving partials of x_t. This expression can be given a nice, intuitive interpretation within the context of harvesting a renewable resource and we postpone its discussion till then. For now it is to be regarded as an equation defining how the multiplier must optimally change through time.

Equation (1.39) is simply a restatement of the difference equation for the state variable and Equations (1.40) and (1.41) are referred to as *boundary conditions* defining the terminal value of the multiplier sequence (λ_T) and the initial condition on the state variable. Because one condition is an initial condition and the other is a terminal condition, the boundary conditions are described as "split."[6]

Collectively, Equations (1.37)–(1.41) form a system of $(3T+1)$ equations in $(3T+1)$ unknowns: y_t for $t = 0, 1, \ldots, T-1$; x_t for $t = 0, 1, \ldots, T$; and λ_t for $t = 1, \ldots, T$. It may be possible to solve the system simultaneously for y_t, x_t, and λ_t although the structure for a particular problem may suggest a more efficient solution algorithm than treating it as a fully simultaneous system. If x_t, y_t, and λ_t are restricted to being nonnegative one must formulate the appropriate Kuhn–Tucker conditions and a solution might be obtained via a nonlinear programming, gradient-based algorithm.

One way of classifying dynamic problems is on the basis of whether terminal time and terminal state are given (fixed) or free to be chosen. From this perspective problem (1.31) would be classified as a "fixed-time, free-state" problem because the horizon was specified but the terminal state was not. In a free-time problem the decision-maker must determine the optimal horizon (i.e., solve for the optimal T).[7] A "restricted free-time" problem may impose a constraint on the length of horizon (e.g., $\underline{t} \le T^* \le \bar{t}$ where $\underline{t}$ and $\bar{t}$ are given). An *infinite* horizon problem, where $T \to \infty$, begs the question of whether or not the solution variables might converge to a set of values and remain unchanged thereafter. Such a solution is referred to as a *steady* or *stationary* state. If in an infinite horizon problem a steady state is attained in period τ then

$$y_t = y^*, \quad x_t = x^*, \quad \text{and } \lambda_t = \lambda^* \text{ for all } t \ge \tau \tag{1.42}$$

The solution to finite (fixed) horizon problems may also lead to a stationary state. For example, it may be optimal for the manager of a mine to deplete his reserves before the end of a given planning horizon. Finally, a "terminal surface" might be specified giving the decision-maker some freedom in the selection of T and x_T, in that he must choose from permissible combinations given by $\phi(T, x_T) = 0$.

---

[6] It may seem to be a minor technicality that, whereas the state variable x_t is specified *initially* by Eq. (1.41), the multiplier λ_t is specified *terminally* by Eq. (1.40). However, this observation is a basic feature of dynamic optimization problems. If λ_t could also be specified initially, the system (1.39)–(1.41) could be completely solved by numerical iteration starting at $t = 0$. The fact that this cannot be done is what makes dynamic optimization difficult—and interesting!

[7] In continuous-time the optimal horizon might be determined by a differential condition $\partial L/\partial T = 0$. In discrete-time there would be no differential relationship and the decision-maker would have to explore horizons of different length, determine the optimal behavior for each horizon (T), and then compare the sum of net economic returns.

Space precludes an exhaustive discussion of the nuances of these terminal conditions. The reader is referred to Kamien and Schwartz (1981) for additional detail.

Because of its importance in many resource management situations we would like to examine more closely the infinite horizon problem and the concept of steady state. Consider the problem

$$\begin{aligned} &\underset{\{y_t\}}{\text{maximize}} \quad \sum_{t=0}^{\infty} V(x_t, y_t) \\ &\text{subject to} \quad x_{t+1} - x_t = f(x_t, y_t) \\ &\qquad x_0 = a \quad \text{given} \end{aligned} \tag{1.43}$$

In contrast to problem (1.31) the above presumes that the objective function has no explicit time dependence (t is not an argument of $V(\cdot)$) and since $T \to \infty$ there is no final function.

The Lagrangian becomes

$$L = \sum_{t=0}^{\infty} \{V(\cdot) + \lambda_{t+1}(x_t + f(\cdot) - x_{t+1})\} \tag{1.44}$$

with first order necessary conditions including:

$$\frac{\partial V(\cdot)}{\partial y_t} + \lambda_{t+1}\frac{\partial f(\cdot)}{\partial y_t} = 0 \tag{1.45}$$

$$\lambda_{t+1} - \lambda_t = -\left(\frac{\partial V(\cdot)}{\partial x_t} + \lambda_{t+1}\frac{\partial f(\cdot)}{\partial x_t}\right) \tag{1.46}$$

$$x_{t+1} - x_t = f(\cdot) \tag{1.47}$$

which must hold for $t = 0, 1, \ldots$. In steady state, with unchanging values for y_t, x_t, and λ_t, Equations (1.45)–(1.47) become a three equation system

$$\frac{\partial V(\cdot)}{\partial y} + \lambda\frac{\partial f(\cdot)}{\partial y} = 0 \tag{1.48}$$

$$\frac{\partial V(\cdot)}{\partial x} + \lambda\frac{\partial f(\cdot)}{\partial x} = 0 \tag{1.49}$$

$$f(\cdot) = 0 \tag{1.50}$$

which might be solved for the steady-state optimum y^*, x^*, and λ^*. By eliminating λ from Equations (1.48) and (1.49) and solving (1.50) for y as a function of x it is often possible to obtain a single equation in the variable x^*.

If a steady-state optimum exists for an infinite horizon problem, if it is unique, and can be found from (1.48)–(1.50), then one might ask: "If we are currently not at the steady-state optimum (i.e., $x_0 \neq x^*$), what is the best way to get there?" There are essentially two types of optimal approach paths from x_0 to x^*, assuming x^* is *reachable* from x_0. The first type is an asymptotic approach in which $x_t \to x^*$ as $t \to \infty$. The second type is called the most rapid approach path (MRAP) in which case x_t is driven to x^* as rapidly as possible, usually reaching x^* in finite time. To drive x_t to x^* as rapidly as possible will often involve a "bang-bang" control where y_t, during the MRAP assumes some maximum or minimum value.

Spence and Starrett (1975) have identified the conditions under which MRAP is optimal. The conditions for problem (1.43) are that (a) via constraint-substitution $V(x_t, y_t)$ must be expressed as an additively separable function in x_t and x_{t+1} and (b) via proper indexing, the problem may be made equivalent to optimization of $\sum_{t=1}^{\infty} w(x_t)$, where $w(\cdot)$ is quasi-concave. Interestingly enough, there are many intuitive specifications for dynamic problems which satisfy the necessary and sufficiency conditions for MRAP to be optimal. If these conditions are met, the solution of the "bang-bang" approach is a relatively trivial matter. We will give an example of such a case, shortly. Before doing so it is appropriate to introduce a more modern control theory concept: the Hamiltonian.

Look closely at conditions (1.37) to (1.39). These conditions define the dynamics between the boundary points. The *Hamiltonian* is defined as

$$\mathcal{H}(x_t, y_t, \lambda_{t+1}, t) = V(x_t, y_t, t) + \lambda_{t+1} f(x_t, y_t) \tag{1.51}$$

and it is possible to write the first order necessary conditions directly as partials of the Hamiltonian. First, note that the Lagrangian expression (1.32) may be written in terms of the Hamiltonian:

$$L = \sum_{t=0}^{T-1} \{\mathcal{H}(\cdot) + \lambda_{t+1}[x_t - x_{t+1}]\} + F(\cdot) \tag{1.52}$$

Then the first order conditions become

$$\frac{\partial L}{\partial y_t} = \frac{\partial \mathcal{H}(\cdot)}{\partial y_t} = 0 \qquad t = 0, \ldots, T-1 \tag{1.53}$$

$$\frac{\partial L}{\partial x_t} = \frac{\partial \mathcal{H}(\cdot)}{\partial x_t} + \lambda_{t+1} - \lambda_t = 0 \qquad t = 1, \ldots, T-1 \tag{1.54}$$

$$\frac{\partial L}{\partial x_T} = -\lambda_T + F'(\cdot) = 0 \tag{1.55}$$

$$\frac{\partial L}{\partial \lambda_{t+1}} = \frac{\partial \mathcal{H}(\cdot)}{\partial \lambda_{t+1}} + x_t - x_{t+1} = 0 \qquad t = 0, \ldots, T-1 \tag{1.56}$$

In their most familiar form these conditions are written as the set

$$\begin{gathered}\frac{\partial \mathcal{H}(\cdot)}{\partial y_t} = 0 \qquad \lambda_{t+1} - \lambda_t = -\frac{\partial \mathcal{H}(\cdot)}{\partial x_t} \qquad x_{t+1} - x_t = \frac{\partial \mathcal{H}(\cdot)}{\partial \lambda_{t+1}} \\ \lambda_T = F'(\cdot) \qquad\qquad x_0 = a\end{gathered} \tag{1.57}$$

The original problem, stated in (1.31), is an example of a subclass of control problems called *open-loop problems*. The solution of such a problem is a control trajectory $\{y_t^*\}$ determined as a function of time, or in our discrete-time problem, in tabular form. Knowing $\{y_t^*\}$ and x_0 one can use the difference equation $x_{t+1} = x_t + f(\cdot)$ to solve forward for the optimal trajectory x_t, denoted $\{x_t^*\}$.

Consider the following problem. As manager of a mine, you are asked to determine the optimal production schedule $\{y_t^*\}$ for $t = 0, \ldots, 9$. The mine is to be shut down and abandoned at $t = 10$. The price per unit of ore is given as $p = 1$ and the cost of extracting y_t is $c_t = y_t^2/x_t$, where x_t is *remaining reserves* at the beginning of period t.

Net revenue may be written as $\pi_t = py_t - y_t^2/x_t = [1 - y_t/x_t]y_t$ and the difference equation describing the change in remaining reserves is $x_{t+1} - x_t = -y_t$, where initial reserves are assumed given with $x_0 = 1{,}000$. Maximization of the sum of net revenues subject to reserve dynamics leads to the Hamiltonian:

$$\mathcal{H}(\cdot) = [1 - y_t/x_t]\, y_t - \lambda_{t+1} y_t$$

with the first order necessary conditions requiring:

$$\begin{aligned}\frac{\partial \mathcal{H}(\cdot)}{\partial y_t} &= 1 - 2y_t/x_t - \lambda_{t+1} = 0 && t = 0, \ldots, 9 \\ \lambda_{t+1} - \lambda_t &= -\frac{\partial \mathcal{H}(\cdot)}{\partial x_t} = -y_t^2/x_t^2 && t = 1, \ldots, 9 \\ x_{t+1} - x_t &= -y_t && t = 0, \ldots, 9 \\ x_0 &= 1{,}000, \quad \lambda_{10} = F'(\cdot) = 0\end{aligned}$$

In this problem there is no final function and any units of x remaining in period 10 must be worthless. Note that this is a fixed-time free-state prob-

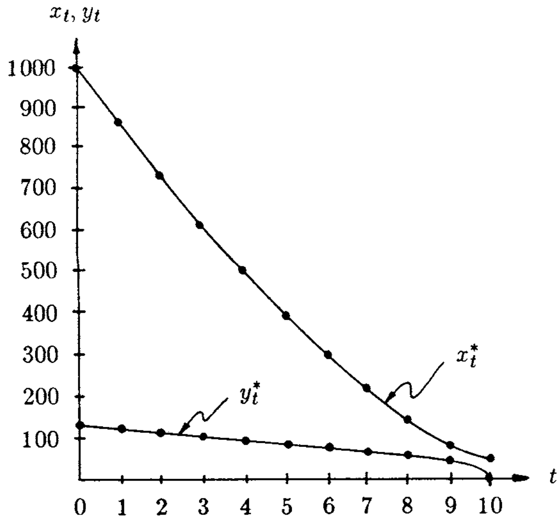
```
10 REM PROGRAM 1.1: MINE PROBLEM
20 DIM X(11),Y(11),Z(11),L(11)
30 L(10) = 0
40 FOR T = 9 TO 0 STEP -1
50   Z(T) = (1 - L(T+1)) / 2
60   L(T) = L(T+1) + Z(T)^2
70 NEXT T
80 X(0) = 1000
90 FOR T = 0 TO 9
100   Y(T) = X(T) * Z(T)
110   X(T+1) = X(T) - Y(T)
120 NEXT T
130 LPRINT " T              X(T)            Y(T)            L(T)"
140 LPRINT "-------------------------------------------------------"
150 LPRINT 0,X(0),Y(0)
160 FOR T = 1 TO 10
170   LPRINT T,X(T),Y(T),L(T)
180 NEXT T
190 END
```

T	X(T)	Y(T)	L(T)
0	1000	138.9018	
1	861.0982	129.3185	.7221965
2	731.7798	119.6851	.6996428
3	612.0947	109.993	.6728931
4	502.1016	100.2317	.6406012
5	401.8699	90.38798	.6007513
6	311.482	80.44653	.550163
7	231.0354	70.39361	.4834595
8	160.6418	60.24068	.390625
9	100.4011	50.20057	.25
10	50.20057	0	0

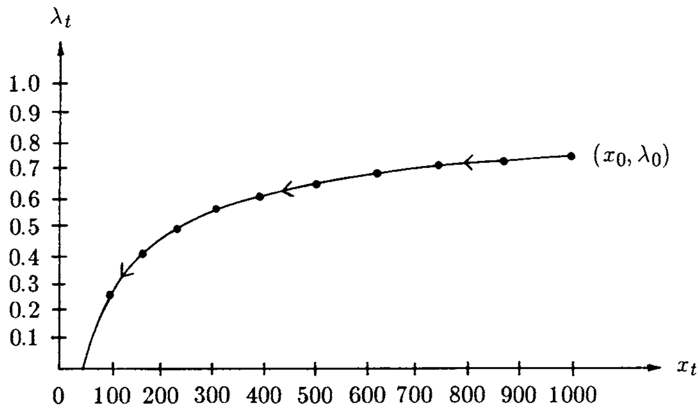
Program 1.1 Solution and algorithm to the mine manager's problem.

lem and that the first order conditions represent a system of 31 equations in 31 unknowns: y_t for $t = 0, 1, \ldots, 9$, x_t for $t = 0, 1, \ldots, 10$, and λ_t for $t = 1, 2, \ldots, 10$. Solution of this problem is most easily accomplished by defining $z_t = y_t/x_t$. Evaluating the $\partial\mathcal{H}(\cdot)/\partial y_t$ at $t = 9$ implies $z_9 = 0.5$ (since $\lambda_{10} = 0$). Evaluating the expression for $\lambda_{t+1} - \lambda_t$ at $t = 9$ implies $\lambda_9 = (z_9)^2 = 0.25$. Knowing λ_9 we can return to $\partial\mathcal{H}(\cdot)/\partial y_t$ to solve for z_8, then back down to the second equation for λ_8, and so forth. The last step in the recursion gives us $z_0 = 0.1389$ and $\lambda_0 = 0.7415$. Knowing that $x_0 = 1000$ we can solve for $y_0 = x_0 z_0 = 138.90$ and $x_1 = x_0 - y_0 = 861.10$. Knowing x_1 we can solve for $y_1 = x_1 z_1 = 129.32$, $x_2 = x_1 - y_1 = 731.78$, and so forth. A solution algorithm (programmed in BASIC) and the complete results are given in Program 1.1.

The optimal time paths $\{y_t^*\}$ and $\{x_t^*\}$ are plotted in Figure 1.4(a), while a plot of the point (x_t^*, λ_t^*) is shown in Figure 1.4(b). The latter



(a) Time paths $\{y_t^*\}$ and $\{x_t^*\}$



(b) Phase plane diagram of x_t^*, λ_t^*

Figure 1.4 Optimal time paths and a phase plane diagram for the mine manager's problem.

graph is referred to as a *phase plane diagram.* Arrows indicate the movement of (x_t^*, λ_t^*) over time. This simple problem can be used to illustrate other aspects of dynamic optimization problems in general and exhaustible resources in particular. We will return to this problem once more in this chapter and again in Chapter 3. We now turn to another important technique for solving dynamic optimization problems.